Journal of Applied Sciences and Modelling
https://journal.krrishpub.com/index.php/jasm/index

Ensuring Stability in Mechanical Contact and Friction Using Linear Complementarity and Matrix Classes

Sunil Kumar [a,*], Monika [b], Rakib Efendiev [c]

[a] Department of Mathematics, School of Sciences, National Institute of Technology Andhra Pradesh, Tadepalligudem, 534 101, India.
[b] Department of Mathematics, Central University of Haryana, Jant-Pali, 123031 Haryana, India.
[c] Mathematics and Computer Science, Baku Engineering University, Baku, AZ 0101, Azerbaijan.

Abstract

Mechanical systems involving unilateral contact and Coulomb friction exhibit nonsmooth, switching behavior driven by inequality constraints. These features challenge classical linear models and may result in nonphysical penetration, negative contact forces, or solver divergence. The linear complementarity problem (LCP) provides a rigorous representation of contact states through mutually exclusive force–gap relationships. This paper studies stability and solvability guarantees for mechanical contact systems through matrix classes associated with LCPs, including semimonotone matrices, Q- and R_0-matrices, and Z-matrices, together with a game-theoretic interpretation of stability. The analysis explains when equilibrium exists and when uniqueness and robustness can be expected. A detailed worked example demonstrates the application of these theoretical tools in evaluating contact stability prior to simulation. The results offer practical guidance for computational mechanics, structural modeling, and contact simulation frameworks.

Keywords: Linear complementarity problem, Contact mechanics, Friction modeling, Matrix-class theory, Game-theoretic stability, Structural equilibrium.

Article Information:
DOI: https://doi.org/10.71426/jasm.v1.i1.pp19-23
Received: *29 Nov. 2025* | Revised: *22 Dec. 2025* | Accepted: *29 Dec. 2025* | Published: *31 Dec. 2025*

1. Introduction

Mechanical contact phenomena arise throughout engineering practice, including robotic manipulation, bolted and jointed structural systems, biomechanical interfaces, particulate media, and machine components subjected to intermittent loading [1], [2], [6]. These systems are governed by unilateral constraints and conditional laws, notably: (i) contact forces are compressive only, (ii) interpenetration is forbidden, and (iii) friction is activated under relative tangential motion [5], [4].

Such behavior is intrinsically nonsmooth and is therefore not adequately captured by purely smooth, unconstrained linear models [5]. A mathematically natural framework is provided by complementarity theory, where admissible states satisfy mutually exclusive conditions [4], [3].

The linear complementarity problem (LCP) seeks vectors $x, w \in \mathbb{R}^n$ such that (1)–(3).

$$x \geq 0, \tag{1}$$
$$w = Ax + q \geq 0, \tag{2}$$
$$x^{\top} w = 0. \tag{3}$$

In (2), the matrix $A \in \mathbb{R}^{n\times n}$ models contact interaction (e.g., stiffness coupling) and $q \in \mathbb{R}^n$ represents external loading and pre-contact geometric effects. The complementarity condition (3) enforces the mutually exclusive nature of active contact reaction ($x_i > 0$) and positive gap ($w_i > 0$).

From an engineering perspective, three questions are central:

- Does a mechanically admissible equilibrium exist for the given loading?
- Is the equilibrium configuration unique?
- Can stability be assessed prior to computation?

This paper demonstrates how matrix-class properties of A provide systematic and verifiable answers to these questions, and how game-theoretic concepts yield an interpretable stability criterion.

*Corresponding author
Email address: sunny061193@gmail.com (Sunil Kumar), yadavmonika1532@gmail.com (Monika), refendiyev@beu.edu.az (Rakib Efendiev).

Nomenclature

Symbol	Description
$x \in \mathbb{R}^n$	Vector of unknown contact reaction forces in the LCP formulation.
$w \in \mathbb{R}^n$	Complementary vector representing contact gaps or residual displacements.
$A \in \mathbb{R}^{n \times n}$	Contact interaction (coupling/stiffness) matrix between contact nodes.
$q \in \mathbb{R}^n$	External loading and pre-contact geometry contribution vector.
N	Normal contact force (compressive-only).
g	Normal geometric gap between contact surfaces.
K	Linearized contact stiffness matrix in unilateral normal contact.
p	Pre-contact displacement/load contribution in the linear gap relation.
T	Tangential traction (frictional force component).
μ	Coulomb friction coefficient.
s	Slip (tangential relative) velocity magnitude.
Δ_n	Probability simplex $\Delta_n = \{z \in \mathbb{R}^n \mid z \geq 0,\ \mathbf{1}^\mathsf{T} z = 1\}$.
$\mathbf{1}$	Vector of ones in $\mathbb{R}^n$.
$v(A)$	Value of the associated two-person zero-sum matrix game (see (17)).
λ_i	Eigenvalues of A.
$x^\mathsf{T} Ax$	Quadratic form representing internal energetic coupling/strain-like interaction.
SPD	Symmetric positive definite matrix: $x^\mathsf{T} Ax > 0$ for all $x \neq 0$.
P-matrix	Matrix with all principal minors positive; implies unique LCP solution for all q.
Q-matrix	Matrix for which LCP(q, A) has at least one solution for all q.
R_0-matrix	Matrix for which LCP$(0, A)$ has only the trivial solution $x = 0$.
Z-matrix	Matrix with nonpositive off-diagonal entries $a_{ij} \leq 0$ for $i \neq j$.

2. Contact and friction as complementarity problems

2.1. Unilateral normal contact

Let N denote the vector of normal contact forces and g the vector of normal gaps. The unilateral (Signorini) contact laws are (4)–(6):

$$N \geq 0, \tag{4}$$

$$g \geq 0, \tag{5}$$

$$N^\mathsf{T} g = 0. \tag{6}$$

Equations (4)–(6) express compressive-only reaction, non-penetration, and mutual exclusivity of force and gap, respectively.

A standard local linearization around an equilibrium configuration yields the affine gap relation (7):

$$g = KN + p, \tag{7}$$

In (7), K is a (linearized) contact stiffness/coupling matrix and p encodes pre-contact geometry and loading effects [16], [15]. Substituting (7) into (4)–(6) yields an LCP of the form (1)–(3).

2.2. Frictional interactions

Coulomb friction constrains tangential traction T by (8).

$$\|T\| \leq \mu N, \tag{8}$$

In (8), μ is the coefficient of friction and N is the normal contact force magnitude. A commonly used complementarity formulation introduces a slip measure $s \geq 0$ and enforces stick–slip complementarity via (9)–(11).

$$\|T\| - \mu N \leq 0, \tag{9}$$

$$s \geq 0, \tag{10}$$

$$(\|T\| - \mu N)s = 0. \tag{11}$$

Equations (9)–(11) encode that either the friction bound is inactive (stick) or the friction bound is active (slip). In practice, combined normal and tangential conditions are frequently reformulated as LCPs or mixed complementarity problems [10], [7], [15], [16].

3. Matrix classes and their mechanical meaning

The solvability and qualitative behavior of (1)–(3) depend critically on the matrix class of A [3], [4]. The classes below are particularly relevant in contact mechanics, where coupling is often sparse and sign-structured.

3.1. P-matrices

A matrix A is a P-matrix if all principal minors are positive. A useful equivalent condition is as follows (12).

$$x \neq 0 \ \Rightarrow\ \exists k \text{ such that } x_k(Ax)_k > 0. \tag{12}$$

P-matrices are fundamental because they guarantee well-posedness:

Theorem (P-matrix LCP well-posedness). If A is a P-matrix, then $\mathrm{LCP}(q, A)$ admits a unique solution for every q [3], [4].

In particular, any symmetric positive definite (SPD) matrix satisfies

$$x^{\mathsf{T}} A x > 0 \quad \forall x \neq 0, \tag{13}$$

and hence is a P-matrix [3].

3.2. Q-matrices

A matrix A is a Q-matrix if $\mathrm{LCP}(q, A)$ has at least one solution for every q [8], [3]. Every P-matrix is a Q-matrix, so (13) implies global solvability of the associated LCP.

3.3. R_0-matrices

A matrix A is an R_0-matrix if the homogeneous problem $\mathrm{LCP}(0, A)$ admits only the trivial solution. Equivalently,

$$x \geq 0,\ Ax \geq 0,\ x^{\mathsf{T}} A x = 0 \ \Rightarrow\ x = 0. \tag{14}$$

Mechanically, (14) rules out spurious residual reaction states in the absence of external loading.

3.4. Z-matrices

A Z-matrix satisfies (15).

$$a_{ij} \leq 0 \quad \text{for } i \neq j. \tag{15}$$

In contact applications, (15) corresponds to purely compressive coupling between contact nodes, with no artificial "tensile" influence across distinct contact degrees of freedom [9], [3].

3.5. Hierarchy

The following implications as expressed in (16) summarize the principal relationships used in this paper.

$$\text{SPD} \Rightarrow \text{P} \Rightarrow \text{Q}, \qquad \text{SPD} \Rightarrow R_0. \tag{16}$$

Accordingly, if A is SPD then $\mathrm{LCP}(q, A)$ is globally solvable and unique for all loadings, a desirable property in equilibrium contact computations.

4. Game-theoretic stability interpretation

Beyond existence and uniqueness, an energetic notion of stability can be expressed through a two-person zero-sum matrix game defined by the payoff matrix A. Define the game value by (17).

$$v(A) = \min_{x \in \Delta_n} \max_{y \in \Delta_n} x^{\mathsf{T}} A y, \tag{17}$$

In (17), the probability simplex is as follows (18).

$$\Delta_n = \{ z \in \mathbb{R}^n \mid z \geq 0,\ \mathbf{1}^{\mathsf{T}} z = 1 \}. \tag{18}$$

In (17), the "structure" chooses x to minimize response while the "environment" chooses y to maximize destabilizing effect. If the value satisfies (19),

$$v(A) > 0, \tag{19}$$

then the system possesses an intrinsic energetic margin: no admissible redistribution of strategies can eliminate resistance.

A closely related sufficient condition is strict copositivity (20):

$$x^{\mathsf{T}} A x > 0 \quad \forall x \geq 0,\ x \neq 0. \tag{20}$$

Condition (20) implies (19) and supports complementarity solvability and stability-type conclusions; see related developments in [13], [14].

5. Applied example: Two-point contact system

5.1. Problem formulation

Consider two contact nodes supporting a rigid body. The associated LCP data are (21):

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \qquad q = \begin{pmatrix} -1 \\ -2 \end{pmatrix}. \tag{21}$$

We seek $x, w \in \mathbb{R}^2$ satisfying (22)-(24).

$$x \geq 0, \tag{22}$$

$$w = Ax + q \geq 0, \tag{23}$$

$$x^{\mathsf{T}} w = 0. \tag{24}$$

5.2. Solvability verification

The matrix A in (21) is symmetric. Its eigenvalues are

$$\lambda_1 = 1, \qquad \lambda_2 = 3, \tag{25}$$

hence A is SPD and therefore a P-matrix by (13). Consequently, $\mathrm{LCP}(q, A)$ admits a unique solution.

5.3. Computation of equilibrium

From (23),

$$w = \begin{pmatrix} 2x_1 - x_2 - 1 \\ -x_1 + 2x_2 - 2 \end{pmatrix}. \tag{26}$$

Assuming both contacts are active ($x_1 > 0$, $x_2 > 0$), complementarity (24) implies $w = 0$, i.e.,

$$\begin{cases} 2x_1 - x_2 = 1, \\ -x_1 + 2x_2 = 2. \end{cases} \tag{27}$$

Solving (27) gives (28).

$$x_1 = \frac{4}{3}, \qquad x_2 = \frac{5}{3}, \tag{28}$$

and substitution into (26) yields

$$w = (0, 0)^{\mathsf{T}}. \tag{29}$$

Thus (22)–(24) hold and the equilibrium is mechanically admissible and unique.

5.4. Game value interpretation

The associated game value (17) (with $n = 2$) is attained at the uniform strategies (30)

$$x^* = y^* = \left(\tfrac{1}{2}, \tfrac{1}{2}\right), \tag{30}$$

and the value is (31).

$$v(A) = (x^*)^\top A y^* = \tfrac{1}{2} > 0. \tag{31}$$

The positivity in (31) is consistent with intrinsic energetic resistance and aligns with the unique admissible LCP solution.

6. Three-node coupled contact system with one detached node

Consider the LCP

$$x \geq 0, \tag{32}$$
$$w = Ax + q \geq 0, \tag{33}$$
$$x^\top w = 0, \tag{34}$$

with

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}, \qquad q = \begin{pmatrix} -2 \\ 4 \\ -4 \end{pmatrix}. \tag{35}$$

6.1. Matrix classification and solvability

The matrix in (35) is symmetric tridiagonal and SPD; its eigenvalues are (36).

$$\lambda_1 = 1, \qquad \lambda_2 = 2, \qquad \lambda_3 = 3. \tag{36}$$

Therefore A is a P-matrix and hence a Q-matrix, implying global solvability. Moreover, SPD implies R_0, excluding spurious unloaded equilibria. Related characterizations for semimonotone, R_0, and Q-matrices are discussed in [11], [8], [9].

6.2. Computation of equilibrium

From (33),

$$w = \begin{pmatrix} 2x_1 - x_2 - 2 \\ -x_1 + 2x_2 - x_3 + 4 \\ -x_2 + 2x_3 - 4 \end{pmatrix}. \tag{37}$$

The coupling pattern suggests possible detachment at the middle node; assume (38)

$$x_2 = 0. \tag{38}$$

Then (37) reduces to

$$w_1 = 2x_1 - 2, \tag{39}$$
$$w_2 = -x_1 - x_3 + 4, \tag{40}$$
$$w_3 = 2x_3 - 4. \tag{41}$$

Assuming nodes 1 and 3 are active ($x_1 > 0$, $x_3 > 0$), complementarity (34) implies (42)

$$w_1 = 0, \qquad w_3 = 0, \tag{42}$$

and from (39) and (41) we obtain

$$x_1 = 1, \qquad x_3 = 2. \tag{43}$$

Substituting (43) into (40) gives (44).

$$w_2 = -1 - 2 + 4 = 1 > 0, \tag{44}$$

which is consistent with detachment at node 2. The complete solution is therefore

$$x = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \qquad w = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}. \tag{45}$$

This configuration satisfies (32)–(34) and is mechanically admissible: contacts at nodes 1 and 3 carry compressive reactions, while node 2 separates with a positive gap.

6.3. Game value interpretation

For $n = 3$, the game value (17) is evaluated at the uniform strategies (46).

$$x^* = y^* = \left(\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}\right)^\top. \tag{46}$$

A direct computation yields (47)

$$Ay^* = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}, \tag{47}$$

and thus

$$v(A) = (x^*)^\top A y^* = \frac{1}{3} > 0. \tag{48}$$

The strict positivity in (48) supports intrinsic energetic resistance and is consistent with the stable contact configuration (45).

7. Engineering implications

The preceding analysis motivates a practical diagnostic workflow for contact computations:

1. Construct the contact interaction matrix A (and load vector q) from the mechanical/FE model.
2. Classify A using verifiable matrix properties (SPD, P, Q, R_0, Z-structure), see (16) and (15).
3. Use these properties to guarantee existence/uniqueness (P, Q, R_0) and to assess energetic resistance via $v(A)$ in (17)–(19).
4. Proceed to numerical simulation only when algebraic guarantees support physically meaningful solutions.

This reduces computational failures (e.g., nonconvergence due to ill-posed complementarity) and prevents nonphysical equilibria.

8. Conclusions

Complementarity theory provides a rigorous and practically relevant foundation for modeling unilateral contact and friction. By linking the LCP formulation (1)–(3) with matrix-class characterizations such as P-, Q-, R_0-, and Z-structures, one can obtain *a priori* guarantees of existence, uniqueness, and admissibility. A complementary stability interpretation is obtained via the zero-sum game value (17), where $v(A) > 0$ serves as an energetic resistance margin consistent with admissible complementarity solutions.

Future work will address dynamic impact and sliding regimes, large-scale finite-element assemblies, and higher-order generalizations such as tensor complementarity and Q-tensors [12].

Declarations and Ethical Statements

Conflict of Interest: The authors declare that there is no conflict of interest.
Funding Statement: The authors declare that no specific funding was received for this research.
Artificial Intelligence usage Statement: During the preparation of this manuscript, the authors utilized ChatGPT solely for language refinement and grammatical corrections. The authors carefully reviewed and revised the generated content and take full responsibility for the accuracy, integrity, and originality of the final manuscript.
Availability of Data and Materials: The data and/or materials that support the findings of this study are available from the corresponding author upon reasonable request.
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